

# ON THE ROTATION REQUIRED FOR THE STABILITY OF AN ELONGATED PROJECTILE.

BY

A. G. GREENHILL, M.A.

*(Professor of Mathematics to the Advanced Class of Artillery Officers.)*

SUPPOSE an elongated projectile—for example, in the shape of a prolate spheroid—moving in infinite frictionless homogeneous liquid of density  $\rho$  under no forces.

Let  $O$  be the centre of the spheroid,  $OC$  the axis of figure,  $OA$  and  $OB$  two equatoreal radii at right angles to one another fixed in the body; and let  $u, v, w$  be the component linear velocities of the body parallel to  $OA, OB$ , and  $OC$ ; and  $p, q, r$  the component angular velocities of the solid about the axes  $OA, OB$ , and  $OC$ .

If the velocity  $u$  only had existed, then the kinetic energy of the body and the surrounding liquid would be  $\frac{1}{2}c_{11}u^2$ ; where  $c_{11}$  is a constant depending on the mass and shape of the body and the density of the liquid;  $c_{11}$  is the force which would have to act on the body for the unit of time to produce the unit of velocity parallel to  $OA$ , and is called the effective inertia of the body in the liquid in the direction  $OA$ ; therefore, also,  $c_{11}u$  in this case is the resultant linear momentum of the solid and liquid.

If the density of the surrounding liquid had been zero, then  $c_{11}$  would be equal to the number of units of mass  $M$  in the body; but in consequence of the velocity  $u$  of the body generating a certain state of motion in the surrounding liquid, we shall have

$$c_{11} = M \left( 1 + \frac{\rho}{\sigma} \alpha \right);$$

where  $\alpha$  is a certain constant depending on the shape of the body,  $\sigma$  being the density of the body.

Since the body is of revolution, it follows that if the velocity  $v$  only had existed, the kinetic energy of the system would be  $\frac{1}{2}c_{11}v^2$ , and the momentum  $c_{11}v$ .

If the velocity  $w$  only had existed, the kinetic energy of the system would be  $\frac{1}{2}c_{33}w^2$ , and the momentum  $c_{33}w$ ; where

$$c_{33} = M \left( 1 + \frac{\rho}{\sigma} \gamma \right),$$

and  $c_{33}$  is the effective inertia of the body in the liquid in the direction  $OC$ ,  $\gamma$  being a constant depending on the shape of the body. In an elongated body  $c_{33}$  is obviously less than  $c_{11}$ .

For instance, if the body be a prolate spheroid of semi-axes  $OC = c$ , and  $OA = a$ , it can be proved that

$$a = \frac{A}{A + C}, \quad \gamma = \frac{C}{2A};$$

where

$$C = \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)^2 (c^2 + \lambda)^{\frac{3}{2}}} = \frac{1}{(c^2 - a^2)^{\frac{3}{2}}} \log \frac{c + \sqrt{c^2 - a^2}}{c - \sqrt{c^2 - a^2}} - \frac{2}{c(c^2 - a^2)},$$

$$A = \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)(c^2 + \lambda)^{\frac{3}{2}}} = \frac{c}{a^2(c^2 - a^2)} - \frac{1}{2(c^2 - a^2)^{\frac{3}{2}}} \log \frac{c + \sqrt{c^2 - a^2}}{c - \sqrt{c^2 - a^2}}.$$

If the angular velocities  $p$  and  $q$  only had existed, the kinetic energy of the system would be

$$\frac{1}{2}c_{44}(p^2 + q^2);$$

where  $c_{44}$  is a constant, called the effective moment of inertia of the body in the liquid about an equatorial axis, and is equal to the couple which would have to act round an equatorial axis of the body for the unit of time to generate the unit of angular velocity; also the components of angular momentum about the axes  $OA$  and  $OB$  would be  $c_{41}p$  and  $c_{41}q$ .

In the prolate spheroid it can be proved that

$$c_{44} = M \frac{1}{5}(a^2 + c^2) \left\{ 1 + \frac{\rho}{\sigma} \frac{C - A}{C - A + \frac{c^2 - a^2}{c^2 + a^2}(C + 2A)} \right\}.$$

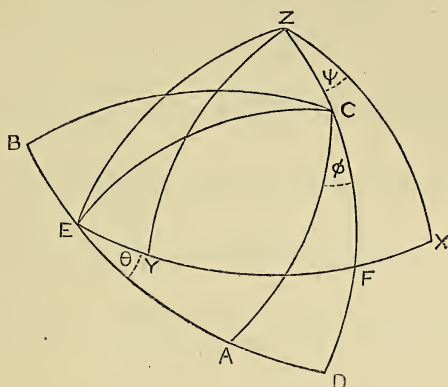
If the angular velocity  $r$  only had existed, then, since this motion of the body will not set the surrounding liquid in motion, the kinetic energy would be  $\frac{1}{2}c_{66}r^2$ ; where  $c_{66}$  is the moment of inertia of the body about its axis of figure  $OC$ ;  $c_{66} = M \frac{2}{5}a^2$ .

Now, if  $T$  denote the kinetic energy of the system when all the velocities  $u, v, w, p, q, r$  co-exist,  $T$  will be equal to the sum of all the separate kinetic energies due to each velocity separately, and therefore

$$T = \frac{1}{2}(c_{11}u^2 + c_{11}v^2 + c_{33}w^2 + c_{44}p^2 + c_{44}q^2 + c_{66}r^2).$$

We must now express  $u, v, w, p, q, r$ , which are estimated with respect to axes fixed in the body, in terms of co-ordinates with respect to lines fixed in space.

Let  $OX, OY, OZ$  be three rectangular axes drawn in fixed directions in space, meeting a sphere described with centre  $O$  and unit radius in



$X, Y, Z$ . (In the figure the eye is supposed to be placed at  $O$ —the centre of the sphere—and to be looking at the concave side of the surface.)

Let the axes  $OA, OB, OC$  of the figure meet the sphere in  $A, B, C$ .

The axes  $XYZ$  may be brought into the position  $ABC$  by a rotation (1) through an angle  $\psi$  suppose, about the axis  $OZ$ , bringing them into the position  $FEZ$ ; (2) by a rotation, through an angle  $\theta$  suppose, about the axis  $OE$ , bringing them into the position  $DEC$ ; and (3) by a rotation, through an angle  $\phi$  suppose, about the axis  $OC$ , bringing them into the position  $ABC$ .

Let  $\dot{x}, \dot{y}, \dot{z}$  denote the component velocities of the point  $O$  parallel to  $OX, OY, OZ$ . We must now express  $u, v, w, p, q, r$  in terms of  $\dot{x}, \dot{y}, \dot{z}, \theta, \phi, \psi$ , and apply Lagrange's equations of motion,

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{x}} \right) - \frac{\delta T}{\delta x} = 0, \dots\dots\dots(1)$$

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{y}} \right) - \frac{\delta T}{\delta y} = 0, \dots\dots\dots(2)$$

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{z}} \right) - \frac{\delta T}{\delta z} = 0, \dots\dots\dots(3)$$

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{\theta}} \right) - \frac{\delta T}{\delta \theta} = 0, \dots\dots\dots(4)$$

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{\phi}} \right) - \frac{\delta T}{\delta \phi} = 0, \dots\dots\dots(5)$$

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{\psi}} \right) - \frac{\delta T}{\delta \psi} = 0, \dots\dots\dots(6)$$

Since  $T$  does not contain  $x, y, z$ , therefore

$$\frac{\delta T}{\delta x} = 0, \quad \frac{\delta T}{\delta y} = 0, \quad \frac{\delta T}{\delta z} = 0;$$

and therefore equations (1), (2), (3) become

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{x}} \right) = 0, \quad \frac{d}{dt} \left( \frac{\delta T}{\delta \dot{y}} \right) = 0, \quad \frac{d}{dt} \left( \frac{\delta T}{\delta \dot{z}} \right) = 0.$$

The integrals of these equations may be written

$$\frac{\delta T}{\delta \dot{x}} = 0, \quad \frac{\delta T}{\delta \dot{y}} = 0, \quad \frac{\delta T}{\delta \dot{z}} = F, \quad \dots\dots\dots(7)$$

a constant, the total momentum of the system, by taking the axis of  $z$  in the direction of the resultant momentum.

Now, the components of linear momentum of the system in the directions  $OA, OB, OC$  are  $c_{11}u, c_{11}v, c_{33}w$ , and therefore

$$c_{11}u = F \cos AZ = -F \sin \theta \cos \phi, \quad \dots\dots\dots(8)$$

$$c_{11}v = F \cos BZ = F \sin \theta \sin \phi, \quad \dots\dots\dots(9)$$

$$c_{33}w = F \cos CZ = F \cos \theta. \quad \dots\dots\dots(10)$$

Now, the component velocities  $\dot{x}, \dot{y}, \dot{z}$  are equivalent to the component velocities  $\dot{x} \cos \psi + \dot{y} \sin \psi, -\dot{x} \sin \psi + \dot{y} \cos \psi, \dot{z}$  in the directions  $OF, OE, OZ$ ; which, again, are equivalent to the component velocities

$$(\dot{x} \cos \psi + \dot{y} \sin \psi) \cos \theta - \dot{z} \sin \theta \text{ in the direction } OD,$$

$$-\dot{x} \sin \psi + \dot{y} \cos \psi \quad \quad \quad \text{ " } \quad \quad \quad OE,$$

$$(\dot{x} \cos \psi + \dot{y} \sin \psi) \sin \theta + \dot{z} \cos \theta \quad \quad \quad \text{ " } \quad \quad \quad OC;$$

which, again, are equivalent to the components

$$\begin{aligned} & \{(\dot{x} \cos \psi + \dot{y} \sin \psi) \cos \theta - \dot{z} \sin \theta\} \cos \phi + (-\dot{x} \sin \psi + \dot{y} \cos \psi) \sin \phi \\ &= (\cos \theta \cos \phi \cos \psi - \sin \phi \sin \psi) \dot{x} + (\cos \theta \cos \phi \sin \psi + \sin \phi \cos \psi) \dot{y} \\ & \quad - \sin \theta \cos \phi \dot{z} \end{aligned}$$

$$= u, \text{ in the direction } OA;$$

$$\begin{aligned} & - \{(\dot{x} \cos \psi + \dot{y} \sin \psi) \cos \theta - \dot{z} \sin \theta\} \sin \phi + (-\dot{x} \sin \psi + \dot{y} \cos \psi) \cos \phi \\ &= -(\cos \theta \sin \phi \cos \psi + \cos \phi \sin \psi) \dot{x} - (\cos \theta \sin \phi \sin \psi - \cos \phi \cos \psi) \dot{y} \\ & \quad + \sin \theta \sin \phi \dot{z} \end{aligned}$$

$$= v, \text{ in the direction } OB; \text{ and}$$

$$(\dot{x} \cos \psi + \dot{y} \sin \psi) \sin \theta + \dot{z} \cos \theta$$

$$= \sin \theta \cos \psi \dot{x} + \sin \theta \sin \psi \dot{y} + \cos \theta \dot{z}$$

$$= w, \text{ in the direction } OC.$$

Conversely, if we express  $\dot{x}, \dot{y}, \dot{z}$  in terms of  $u, v, w, \theta, \phi, \psi$ , we shall have

$$\begin{aligned}\dot{x} \cos \psi + \dot{y} \sin \psi &= \text{component velocity in the direction } OD \\ &= (u \cos \phi - v \sin \phi) \cos \theta + w \sin \theta \\ &= -\frac{F}{c_{11}} (\sin \theta \cos^2 \phi + \sin \theta \sin^2 \phi) \cos \theta + \frac{F}{c_{33}} \sin \theta \cos \theta \\ &= F \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin \theta \cos \theta;\end{aligned}$$

$$\begin{aligned}-\dot{x} \sin \psi + \dot{y} \cos \psi &= \text{component velocity in the direction } OE \\ &= u \sin \phi + v \cos \phi = 0;\end{aligned}$$

$$\text{and therefore } \dot{x} = F \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin \theta \cos \theta \cos \psi, \quad \dots\dots\dots(11)$$

$$\dot{y} = F \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin \theta \cos \theta \sin \psi; \quad \dots\dots\dots(12)$$

$$\begin{aligned}\text{also } \dot{z} &= -u \sin \theta \cos \phi + v \sin \theta \sin \phi + w \cos \theta \\ &= F \left( \frac{\cos^2 \theta}{c_{33}} + \frac{\sin^2 \theta}{c_{11}} \right). \quad \dots\dots\dots(13)\end{aligned}$$

We have therefore expressed  $\dot{x}, \dot{y}, \dot{z}$ —the component velocities of the point  $O$  parallel to the fixed directions  $OX, OY, OZ$ —in terms of  $\theta, \psi$ , and constants, the angle  $\phi$  not appearing; and  $u, v, w$ —the component velocities of the point  $O$ , parallel to the axes  $OA, OB, OC$ , moving in space and fixed in the body—in terms of  $\theta, \phi$ , and constants, the angle  $\psi$  not appearing.

We must now express  $p, q, r$  in terms of  $\theta, \phi, \psi$  and  $\dot{\theta}, \dot{\phi}, \dot{\psi}$ —their rates of increase per unit of time.

We have

$$\begin{aligned}p &= \sin \phi \dot{\theta} - \sin \theta \cos \phi \dot{\psi}, \\ q &= \cos \phi \dot{\theta} + \sin \theta \sin \phi \dot{\psi}, \\ r &= \dot{\phi} + \cos \theta \dot{\psi};\end{aligned}$$

and expressing  $T$  in terms of  $\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}$ ,

$$T = \frac{1}{2} F^2 \left( \frac{\cos^2 \theta}{c_{33}} + \frac{\sin^2 \theta}{c_{11}} \right) + \frac{1}{2} c_{44} (\dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2) + \frac{1}{2} c_{66} (\dot{\phi} + \cos \theta \dot{\psi})^2;$$

$$\text{and therefore } \frac{\delta T}{\delta \phi} = 0, \quad \frac{\delta T}{\delta \psi} = 0.$$

To determine  $\theta$ ,  $\phi$ ,  $\psi$  as functions of  $t$ , we have by equations (4), (5) and (6), and first from equation (5),

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{\phi}} \right) = 0,$$

and therefore

$$\frac{\delta T}{\delta \dot{\phi}} = \text{constant},$$

or

$$\dot{\phi} + \cos \theta \dot{\psi} = r,$$

a constant—the angular velocity of the body about the axis  $OC$ , which therefore remains constant during the motion.

Also from (6),

$$\frac{d}{dt} \left( \frac{\delta T}{\delta \dot{\psi}} \right) = 0;$$

therefore

$$\begin{aligned} \frac{\delta T}{\delta \dot{\psi}} &= c_{44} \sin^2 \theta \dot{\psi} + c_{66} (\dot{\phi} + \cos \theta \dot{\psi}) \cos \theta \\ &= c_{44} \sin^2 \theta \dot{\psi} + c_{66} r \cos \theta = G, \quad \dots\dots\dots (14) \end{aligned}$$

a constant—the angular momentum of the system about the axis of  $z$ .

To obtain  $\frac{\delta T}{\delta \theta}$  we must differentiate the expression for  $T$  in terms of  $u$ ,  $v$ ,  $w$ ,  $p$ ,  $q$ ,  $r$ ; and therefore

$$\frac{\delta T}{\delta \theta} = \frac{\delta T}{\delta u} \frac{\delta u}{\delta \theta} + \frac{\delta T}{\delta v} \frac{\delta v}{\delta \theta} + \frac{\delta T}{\delta w} \frac{\delta w}{\delta \theta} + \frac{\delta T}{\delta p} \frac{\delta p}{\delta \theta} + \frac{\delta T}{\delta q} \frac{\delta q}{\delta \theta} + \frac{\delta T}{\delta r} \frac{\delta r}{\delta \theta}.$$

But

$$\frac{\delta u}{\delta \theta} = -w \cos \phi, \quad \frac{\delta v}{\delta \theta} = w \sin \phi,$$

$$\frac{\delta w}{\delta \theta} = u \cos \phi - v \sin \phi;$$

and therefore

$$\begin{aligned} \frac{\delta T}{\delta \theta} &= (c_{33} - c_{11}) (u \cos \phi - v \sin \phi) w \\ &\quad + c_{44} \sin \theta \cos \theta \dot{\psi}^2 - c_{66} r \sin \theta \dot{\psi} \\ &= F^2 \sin \theta \cos \theta \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \\ &\quad + c_{44} \sin \theta \cos \theta \dot{\psi}^2 - c_{66} r \sin \theta \dot{\psi}, \end{aligned}$$

and

$$\frac{\delta T}{\delta \dot{\theta}} = c_{44} \dot{\theta};$$

therefore equation (4) becomes

$$c_{44} \ddot{\theta} - c_{44} \sin \theta \cos \theta \dot{\psi}^2 + c_{66} r \sin \theta \dot{\psi} - F^2 \sin \theta \cos \theta \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) = 0; \quad \dots (15)$$

and since from equation (14), differentiating with respect to  $t$ ,

$$c_{66}' \sin \theta \dot{\theta} = 2c_{44} \sin \theta \cos \theta \dot{\theta} \dot{\psi} + c_{44} \sin^2 \theta \ddot{\psi},$$

therefore

$$c_{44} \ddot{\theta} + c_{44} \sin \theta \cos \theta \dot{\theta} \dot{\psi}^2 + c_{44} \sin^2 \theta \ddot{\psi} - F^2 \sin \theta \cos \theta \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) = 0; \dots (16)$$

and integrating,

$$\frac{1}{2} c_{44} (\dot{\theta}^2 + \sin^2 \theta \dot{\psi}^2) + \frac{1}{2} c_{66}' r^2 + \frac{1}{2} F^2 \left( \frac{\cos^2 \theta}{c_{33}} + \frac{\sin^2 \theta}{c_{11}} \right) = T, \dots (17)$$

a constant. The kinetic energy is therefore constant—a principle we might have assumed.

Eliminating  $\dot{\psi}$  between equations (14) and (17), we shall obtain an equation the integral of which will give  $\theta$  in terms of elliptic functions of  $t$ .

Considering, however, only the case of the steady motion of the body, when the point  $O$  describes a uniform helix whose axis is in the direction of the axis  $OZ$ , the angle  $\theta$  between the axis  $OC$  of the body and  $OZ$  remaining constant; putting  $\theta = a$ , a constant, and  $\dot{\psi} = \mu$ , a constant, and therefore  $\psi = \mu t$ , we have

$$\dot{x} = F \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin a \cos a \cos \mu t,$$

$$\dot{y} = F \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin a \cos a \sin \mu t,$$

$$\dot{z} = F \left( \frac{\cos^2 a}{c_{33}} + \frac{\sin^2 a}{c_{11}} \right);$$

and therefore

$$x = \frac{F}{\mu} \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin a \cos a \sin \mu t,$$

$$y = -\frac{F}{\mu} \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \sin a \cos a \cos \mu t,$$

$$z = F \left( \frac{\cos^2 a}{c_{33}} + \frac{\sin^2 a}{c_{11}} \right) t;$$

the equations of the helix described by the centre  $O$  of the body.

Equation (15) becomes, since  $\ddot{\theta} = 0$ ,

$$c_{44} \sin a \cos a \mu^2 - c_{66}' \sin a \mu + F^2 \sin a \cos a \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) = 0;$$

and dividing out by  $\sin a$ ,

$$c_{44} \cos a \mu^2 - c_{66}' \mu + F^2 \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \cos a = 0.$$

In order that this quadratic equation should give real values of  $\mu$ , we must have

$$\begin{aligned} n^2 &> 4F^2 \cos^2 \alpha \frac{c_{44}}{c_{66}^2} \left( \frac{1}{c_{33}} - \frac{1}{c_{11}} \right) \\ &> 4w^2 \frac{c_{33}c_{44}}{c_{66}^2} \left( 1 - \frac{c_{33}}{c_{11}} \right). \end{aligned}$$

Suppose the projectile fired from a rifled gun in which the rifling at the muzzle would make one turn in  $n$  calibres, and suppose  $\beta$  the angle the rifling at the muzzle makes with the axis of the bore; then

$$\tan \beta = \frac{\pi d}{n d} = \frac{\pi}{n};$$

$d$  being the calibre; also

$$\tan \beta = \frac{\frac{1}{2} dr}{w};$$

$w$  being the muzzle velocity, and  $r$  the angular velocity; therefore

$$n = \frac{2\pi w}{dr}.$$

Hence, if the rifling of the gun required to give the requisite spin make one turn in  $n$  calibres, the calibre being  $d$ , we must have

$$n = \frac{\pi}{d} \sqrt{\frac{c_{11}c_{66}^2}{c_{33}c_{44}(c_{11} - c_{33})}}.$$

Where, as in practice, the fraction  $\frac{\rho}{\sigma}$ , which is equal to the ratio of the weight of the air displaced to the weight of the projectile, is so small that the squares and higher powers may be neglected, then in the formula

$$n^2 = \frac{\pi^2 c_{11} c_{66}^2}{4a^2 c_{33} c_{44} (c_{11} - c_{33})}$$

we may put

$$\frac{c_{11}}{c_{33}} = 1;$$

but

$$c_{11} - c_{33} = M \frac{\rho}{\sigma} (\alpha - \gamma),$$

and

$$c_{44} = M k_1^2, \quad c_{66} = M k_3^2,$$

where  $k_1$  and  $k_3$  are the radii of gyration of the projectile about an equatoreal axis through the centre of gravity and about the axis of figure, and therefore we may put

$$n^2 = \frac{\pi^2 k_3^4}{4a^2 k_1^2 \frac{\rho}{\sigma} (\alpha - \gamma)}.$$

This formula shows, therefore, that  $n$  remains the same for similar projectiles made of the same material, whatever be the calibre.

## NUMERICAL APPLICATIONS.

As numerical examples, consider firstly common unfilled shell, which may, for the purposes of theoretical calculation, be considered as prolate spheroids of which the length is three diameters, the cavity being formed by a similar surface of two-thirds the linear dimensions of the external surface.

Here  $c = 3a$ , and therefore

$$A = \frac{1}{a^3} \left\{ \frac{3}{8} - \frac{1}{32} \sqrt{2} \log_{\epsilon} (3 + \sqrt{8}) \right\},$$

$$C = \frac{1}{a^3} \left\{ \frac{1}{16} \sqrt{2} \log_{\epsilon} (3 + \sqrt{8}) - \frac{1}{12} \right\}.$$

Now,

$$\log \sqrt{8} = 0.4515450,$$

therefore

$$\sqrt{8} = 2.82843,$$

$$3 + \sqrt{8} = 5.82843,$$

$$\log_{10} (3 + \sqrt{8}) = 0.7655493.$$

But

$$\log_{\epsilon} (3 + \sqrt{8}) = \mu \log_{10} (3 + \sqrt{8}),$$

where

$$\mu = 2.30258509,$$

and

$$\log_{10} \mu = 0.3622185,$$

$$\log_{10} \log_{10} (3 + \sqrt{8}) = 1.8839736;$$

therefore

$$\log_{10} \log_{\epsilon} (3 + \sqrt{8}) = 0.2461921,$$

and

$$\log_{\epsilon} (3 + \sqrt{8}) = 1.76275.$$

Also,

$$\log \sqrt{2} = 0.1505150,$$

therefore

$$\log_{10} \sqrt{2} \log_{\epsilon} (3 + \sqrt{8}) = 0.3967071,$$

and

$$\sqrt{2} \log_{\epsilon} (3 + \sqrt{8}) = 2.49291,$$

and

$$\frac{1}{16} \sqrt{2} \log_{\epsilon} (3 + \sqrt{8}) = 0.15581.$$

Therefore

$$a^3 A = \frac{3}{8} - 0.0779$$

$$= 0.3750$$

$$- 0.0779$$

$$= 0.2971;$$

$$a^3 C = 0.15581 - \frac{1}{12}$$

$$= 0.15581$$

$$- 0.08333$$

$$= 0.0725.$$

Therefore

$$a = \frac{A}{A + C} = \frac{2971}{3696},$$

$$\gamma = \frac{C}{2A} = \frac{725}{5942},$$

$$\begin{aligned} \log a &= 3.4729027 \\ &\quad - 3.5677320 \\ &= \bar{1}.9051707, \end{aligned}$$

$$a = 0.80385,$$

$$\begin{aligned} \log \gamma &= 2.8603380 \\ &\quad - 3.7739326 \\ &= \bar{1}.0864054, \end{aligned}$$

$$\gamma = 0.12201,$$

and

$$a - \gamma = 0.68184.$$

Now,

$$\frac{\rho}{\sigma} = \frac{\text{weight of the displaced air}}{\text{weight of the shell}},$$

and the density of cast-iron is 7.207 times the density of water, while the density of air at 15.5° C. or 60° F., the height of the barometer being 30 ins., is  $\frac{1}{816}$  of the density of water; therefore

$$\begin{aligned} \frac{\rho}{\sigma} &= \frac{\frac{4}{3} \pi a^3 c \times \frac{1}{816}}{\frac{4}{3} \pi a^3 c \left(1 - \frac{2^3}{3^3}\right) \times 7.207} \\ &= \frac{27}{19 \times 7.207 \times 816} \end{aligned}$$

$$\log 27 = 1.4313638$$

$$\log 19 = 1.2787536$$

$$\log 7.207 = 0.8577545$$

$$\log 816 = 2.9116902$$

$$\log \frac{\rho}{\sigma} = 1.4313638$$

$$\quad - 5.0481983$$

$$= \bar{4}.3831655,$$

$$\frac{\rho}{\sigma} = 0.00024164,$$

and therefore  $\frac{\rho}{\sigma}$  is a fraction small enough for the squares and higher powers to be neglected; and therefore

$$n^2 = \frac{\pi^2 k_3^4}{4a^2 k_1^2 \frac{\rho}{\sigma} (a - \gamma)};$$

where  $k_1, k_3$  are the equatoreal and polar radii of gyration of the shell, and putting  $\frac{c_{33}}{c_{11}} = 1$ ,  $c_{66} = Mk_3^2$ ,  $c_{44} = Mk_1^2$ ,  $c_{11} - c_{33} = M \frac{\rho}{\sigma} (\alpha - \gamma)$ .

$$\begin{aligned}\text{Now, } k_1^2 &= \frac{\frac{4}{3} \pi a^2 c \cdot \frac{1}{5} (a^2 + c^2) \left(1 - \frac{2^5}{3^5}\right)}{\frac{4}{3} \pi a^2 c \left(1 - \frac{2^3}{3^3}\right)} \\ &= \frac{211}{171} \times \frac{1}{5} (a^2 + c^2) \\ &= \frac{211}{171} \times 2a^2, \text{ if } c = 3a;\end{aligned}$$

$$\begin{aligned}\text{and similarly, } k_3^2 &= \frac{1 - \frac{2^5}{3^5}}{1 - \frac{2^3}{3^3}} \times \frac{2}{5} a^2 \\ &= \frac{211}{171} \times \frac{2}{5} a^2.\end{aligned}$$

$$\begin{aligned}\text{Therefore } n^2 &= \frac{\pi^2 \times \left(\frac{211}{171}\right)^2 \times \frac{4}{25}}{4 \times \frac{211}{171} \times 2 \times \frac{\rho}{\sigma} (\alpha - \gamma)} \\ &= \frac{211\pi^2}{50 \times 171 \times \frac{\rho}{\sigma} (\alpha - \gamma)}\end{aligned}$$

$$\log 211 = 2.3242825,$$

$$\log \pi^2 = 0.9943018;$$

$$\text{therefore } \log 211\pi^2 = 3.3185843,$$

$$\log 50 = 1.6989700,$$

$$\log 171 = 2.2329961,$$

$$\log \frac{\rho}{\sigma} = 4.3831655,$$

$$\begin{aligned}\log (\alpha - \gamma) &= 1.8336825 \\ &= 0.1488141;\end{aligned}$$

$$\text{therefore } \log n^2 = 3.1697702,$$

$$\log n = 1.5848851,$$

$$n = 38.449.$$

In the service guns  $n$  is generally made equal to 35.

Secondly, consider the case of double shell, unfilled, which must, for purposes of theoretical calculation, be considered as prolate spheroids

of which the length is four diameters, and the cavity is a similar surface of two-thirds the linear dimensions of the external surface.

Then  $\frac{\rho}{\sigma}$  has the same value as before—viz., .00024164; but since  $c = 4a$ , therefore

$$a^3 A = \frac{4}{15} - \frac{1}{15^{\frac{3}{2}}} \log_{\epsilon} (4 + \sqrt{15}),$$

$$a^3 C = \frac{2}{15^{\frac{3}{2}}} \log_{\epsilon} (4 + \sqrt{15}) - \frac{1}{30}.$$

Now,  
therefore

$$\log \sqrt{15} = 0.5880456,$$

$$\sqrt{15} = 3.8723,$$

$$4 + \sqrt{15} = 7.8723,$$

$$\log_{10} (4 + \sqrt{15}) = 0.8961403,$$

$$\log_{10} \log_{10} (4 + \sqrt{15}) = 1.9523759,$$

$$\log_{10} \mu = 0.3622185,$$

$$\log_{10} \log_{\epsilon} (4 + \sqrt{15}) = 0.3145944,$$

$$\log 15^{\frac{3}{2}} = 1.7641368,$$

$$\therefore \log_{10} \frac{1}{15^{\frac{3}{2}}} \log_{\epsilon} (4 + \sqrt{15}) = 2.5504576,$$

$$\frac{1}{15^{\frac{3}{2}}} \log_{\epsilon} (4 + \sqrt{15}) = 0.035518.$$

Therefore

$$a^3 A = .266666$$

$$- .035518$$

$$= .231148,$$

$$a^3 C = .071036$$

$$- .033333$$

$$= .037703;$$

$$\alpha = \frac{A}{A + C} = \frac{231148}{268851} = .85977,$$

$$\gamma = \frac{C}{2A} = \frac{37703}{462296} = .08156,$$

$$\alpha - \gamma = .77821.$$

Also, as before,

$$\begin{aligned} k_1^2 &= \frac{211}{171} \times \frac{1}{5} (a^2 + c^2) \\ &= \frac{211}{171} \times \frac{17}{5} a^2, \text{ since } c = 4a; \end{aligned}$$

and

$$k_3^2 = \frac{211}{171} \times \frac{2}{5} a^2.$$

Therefore

$$\begin{aligned} n^2 &= \frac{\pi^2 \times \left(\frac{211}{171}\right)^2 \times \frac{4}{25}}{4 \times \frac{211}{171} \times \frac{17}{5} \times \frac{\rho}{\sigma} \times (a - \gamma)} \\ &= \frac{211\pi^2}{5 \times 171 \times 17 \times \frac{\rho}{\sigma} \times (a - \gamma)}, \end{aligned}$$

$$\log 211\pi^2 = 3.3185843,$$

$$\log 5 = 0.6989700,$$

$$\log 171 = 2.2329961,$$

$$\log 17 = 1.2304489,$$

$$\log \frac{\rho}{\sigma} = 4.3831655,$$

$$\log (a - \gamma) = \frac{1.8910968}{0.4366773};$$

therefore

$$\log n^2 = 2.8819070,$$

$$\log n = 1.4409535;$$

therefore

$$n = 27.603.$$

Consequently, even for double shell, the twist in the 6.3-in. and the 8-in. howitzers—one turn in sixteen calibres—seems excessive; the spin imparted being theoretically far in excess of what is required for stability, the only effect of the excess of spin being to diminish the drift, which increases as the angular velocity diminishes.

When the shell is filled the stability is slightly increased, and the spin need not be so great. For suppose a common shell, filled with powder of density 1.6; then, since  $a$  and  $\gamma$  depend only on the external shape of the shell, they have the same values as before, and

$$a - \gamma = .68184.$$

But

$$\frac{\rho}{\sigma} = \frac{\text{weight of the displaced air}}{\text{weight of the shell}}$$

$$\begin{aligned} &= \frac{\frac{4}{3} \pi a^2 c \times \frac{1}{816}}{\frac{4}{3} \pi a^2 c \left(1 - \frac{2^3}{3^3}\right) \times 7.207 + \frac{4}{3} \pi a^2 c \frac{2^3}{3^3} \times 1.6} \\ &= \frac{27}{(19 \times 7.207 + 8 \times 1.6) 816} \\ &= \frac{27}{149.733 \times 816}, \end{aligned}$$

$$\log 27 = 1.4313638,$$

$$\log 149.733 = 2.1753176,$$

$$\log 816 = 2.9116902,$$

$$\log \frac{\rho}{\sigma} = 1.4313638$$

$$- 5.0870078,$$

$$= \bar{4}.3443560,$$

$$\frac{\rho}{\sigma} = 0.00022098,$$

$$k_1^2 = \frac{\frac{4}{3} \pi a^2 c \times \frac{1}{5} (a^2 + c^2) \left\{ \left(1 - \frac{2^5}{3^5}\right) \times 7.207 + \frac{2^5}{3^5} \times 1.6 \right\}}{\frac{4}{3} \pi a^2 c \left\{ \left(1 - \frac{2^3}{3^3}\right) \times 7.207 + \frac{2^3}{3^3} \times 1.6 \right\}}$$

$$= \frac{211 \times 7.207 + 32 \times 1.6}{19 \times 7.207 + 8 \times 1.6} \times \frac{1}{9} \times \frac{1}{5} (a^2 + c^2)$$

$$= \frac{1571.877}{149.733} \times \frac{1}{9} \times \frac{1}{5} (a^2 + c^2)$$

$$= \frac{1571.877}{149.733} \times \frac{2}{9} a^2, \text{ if } c = 3a;$$

and

$$k_3^2 = \frac{1571.877}{149.733} \times \frac{1}{9} \times \frac{2}{5} a^2.$$

Therefore

$$\begin{aligned} n^2 &= \frac{\pi^2 k_3^4}{4a^2 k_1^2 \frac{\rho}{\sigma} (a - \gamma)} \\ &= \frac{\pi^2 \times 1571.877}{9 \times 50 \times 149.733 \times \frac{\rho}{\sigma} (a - \gamma)} \end{aligned}$$

$$\log \pi^2 = 0.9943018,$$

$$\log 1571.877 = 3.1964249$$

$$= 4.1907267,$$

$$\log 9 = 0.9542425,$$

$$\log 50 = 1.6989700,$$

$$\log 149.733 = 2.1753176,$$

$$\log \frac{\rho}{\sigma} = \bar{4}.3443560,$$

$$\log (a - \gamma) = \bar{1}.8336825$$

$$= 1.0065686$$

therefore

$$\log n^2 = 3.1841581,$$

$$\log n = 1.5920790,$$

$$n = 39.09.$$

Similarly with filled double shell we shall find  $n = 28$ , about.

If the Martini-Henry bullet be considered a solid prolate spheroid of lead, of which the length is three diameters, then

$$\alpha - \gamma = .68184,$$

$$\frac{\rho}{\sigma} = \frac{\text{density of air}}{\text{density of lead}}$$

$$= \frac{1}{\frac{816}{10.9}} = \frac{1}{10.9 \times 816}$$

$$k_1^2 = \frac{1}{5} (a^2 + c^2) = 2a^2,$$

$$k_3^2 = \frac{2}{5} a^2;$$

and therefore

$$n^2 = \frac{\pi^2 \times 10.9 \times 816}{50 \times .68184},$$

$$\log \pi^2 = 0.9943018,$$

$$\log 10.9 = 1.0374265,$$

$$\log 816 = 2.9116902$$

$$= 4.9434185,$$

$$\log 50 = 1.6989700,$$

$$\log .68184 = \bar{1}.8336825,$$

$$= 1.5326525,$$

$$\log n^2 = 3.4107660,$$

$$\log n = 1.7053830,$$

$$n = 50.744.$$

In the Martini-Henry rifle the rifling makes one turn in 22 ins.; and the calibre being .45 of an inch, the rifling therefore makes one turn in  $\frac{22}{.45} = 49$  calibres.

Generally, for a projectile  $x$  calibres long,  $\frac{c}{a} = x$ , and therefore

$$a^3 C = \frac{2}{(x^2 - 1)^{\frac{3}{2}}} \log_e (x + \sqrt{x^2 - 1}) - \frac{2}{x(x^2 - 1)},$$

$$a^3 A = \frac{x}{x^2 - 1} - \frac{1}{(x^2 - 1)^{\frac{3}{2}}} \log_e (x + \sqrt{x^2 - 1});$$

and

$$\begin{aligned}
 \alpha - \gamma &= \frac{A}{A+C} - \frac{C}{2A} \\
 &= \frac{2A^2 - AC - C^2}{2A(A+C)} \\
 &= \frac{2A+C}{2A} \times \frac{A-C}{A+C} \\
 &= \frac{\frac{2}{x}}{\frac{2x}{x^2-1} - \frac{2}{(x^2-1)^{\frac{3}{2}}} \log(x + \sqrt{x^2-1})} \\
 &\quad \times \frac{\frac{x^2+2}{x(x^2-1)} - \frac{3}{(x^2-1)^{\frac{3}{2}}} \log(x + \sqrt{x^2-1})}{\frac{x^2-2}{x(x^2-1)} + \frac{1}{(x^2-1)^{\frac{3}{2}}} \log(x + \sqrt{x^2-1})} \\
 &= \frac{\frac{1}{x}(x^2-1)^{\frac{3}{2}}}{x\sqrt{x^2-1} - \log(x + \sqrt{x^2-1})} \\
 &\quad \times \frac{\left(x + \frac{2}{x}\right)\sqrt{x^2-1} - 3\log(x + \sqrt{x^2-1})}{x - \frac{2}{x}\sqrt{x^2-1} + \log(x + \sqrt{x^2-1})}.
 \end{aligned}$$

If the cavity be a similar spheroid of  $\frac{1}{m}$ th the linear dimensions of the external surface,

$$\begin{aligned}
 k_1^2 &= \frac{\pi a^2 c \frac{1}{5} (a^2 + c^2) \left(1 - \frac{1}{m^5}\right)}{\pi a^2 c \left(1 - \frac{1}{m^3}\right)} \\
 &= \frac{m^5 - 1}{m^5 - m^2} \cdot \frac{1}{5} (a^2 + c^2), \\
 k_3^2 &= \frac{m^5 - 1}{m^5 - m^2} \cdot \frac{2}{5} a^2,
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\rho}{\sigma} &= \frac{\text{weight of shell}}{\text{weight of displaced air}} \\
 &= \left(1 - \frac{1}{m^3}\right) \frac{\text{density of iron}}{\text{density of air}}.
 \end{aligned}$$

Therefore, generally, we must have

$$\begin{aligned}
 n^2 &= \frac{\pi^2 \left(\frac{m^5-1}{m^5-m^2}\right)^2 \frac{4}{25} a^4 \frac{m^3-1}{m^3}}{4a^2 \frac{m^5-1}{m^5-m^2} \frac{1}{5} (a^2+c^2) (\alpha-\gamma)} \times \frac{\text{density of iron}}{\text{density of air}} \\
 &= \frac{1}{5} \pi^2 \left(1 - \frac{1}{m^5}\right) \frac{1}{x^2+1} \times \frac{1}{\alpha-\gamma} \times \frac{\text{density of iron}}{\text{density of air}};
 \end{aligned}$$

$$i.e., \quad \frac{1}{5} \pi^2 \left(1 - \frac{1}{m^5}\right) \frac{1}{x^2 + 1} \times \frac{1}{a - \gamma} \times \frac{\text{density of iron}}{\text{density of air}};$$

where  $a - \gamma$  is given by the formula above, giving  $n$  the number of calibres to one turn for the rifling at the muzzle of a gun firing shell of which the length is  $x$  calibres, and the cavity  $\frac{1}{m}$ th of the linear dimensions of the external surface.

The formula is obtained on the hypotheses that gravity is neglected, that the surrounding medium is frictionless and incompressible, and that the external surface of the shell is a prolate spheroid, and the results obtained agree fairly with what is observed in practice.

For instance, the Krupp 40<sup>cm</sup> gun fires a steel shell 2·8 diameters long, and the rifling at the muzzle makes one turn in 45 calibres.

As the steel shell is to be employed against armour, we may assume that the linear dimensions of the cavity are one-half those of the external surface, as in our Palliser projectiles, and therefore  $m = 2$  and  $x = 2\cdot8$ .

We have

$$\begin{aligned} \sqrt{x^2 - 1} &= 2\cdot61534, \\ x + \sqrt{x^2 - 1} &= 5\cdot41534, \\ \log_e (x + \sqrt{x^2 - 1}) &= 1\cdot68924, \\ \frac{1}{x} (x^2 - 1)^{\frac{3}{2}} &= 6\cdot38890, \\ x \sqrt{x^2 - 1} &= 7\cdot32294, \\ \frac{2}{x} \sqrt{x^2 - 1} &= 1\cdot86810, \\ \left(x + \frac{2}{x}\right) \sqrt{x^2 - 1} &= 9\cdot19104, \\ \left(x - \frac{2}{x}\right) \sqrt{x^2 - 1} &= 5\cdot45484; \end{aligned}$$

and therefore by the formula

$$\begin{aligned} a - \gamma &= \frac{6\cdot38890}{7\cdot32294 - 1\cdot68924} \times \frac{9\cdot19104 - 5\cdot06772}{5\cdot45484 + 1\cdot68924} \\ &= \frac{6\cdot38890}{5\cdot63370} \times \frac{4\cdot12332}{7\cdot14408} = \cdot654535; \end{aligned}$$

and therefore

$$n^2 = \frac{1}{5} \pi^2 \times \frac{31}{32} \times \frac{1}{8\cdot84} \times \frac{1}{\cdot654535} \times 7\cdot8 \times 816;$$

and working this out by logarithms, we shall find  $n = 45\cdot753$ .